

History of Binomial and Multinomial Expansions

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ABSTRACT

In applied mathematics Binomial Expansion and Multinomial expansion are of great importance. In around 300 BCE Indian mathematician Pingala had derived the method of obtaining a triangular arrangement known as “Meru-Prastar” for attainment of coefficients of binomial expansion. In sixteenth century, CE it was rediscovered by French mathematician Blasé Pascal (1588-1688CE) and termed as Pascal’s triangle. This paper discusses the development of binomial expansion, multinomial expansion with its applications. The paper also emphasizes the fact that the historical roots of binomial expansion are embedded in Pingalacharya’s Meru-Prastar.

Keywords: Binomial-Theorem, Meru-Prastar, Varnic-Meru, Multinomial Expansion, Leibnitz Theorem.

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1. INTRODUCTION

Binomial Expansion : By Binomial Expansion we mean the expansion of a binomial into a sum involving terms of the form ax^by^c where the coefficient of each term is a positive integer and sum of exponents of x and y always tends to n [18]. For example:

$$(x+y)^n = x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 + \dots + {}^nC_r x^{n-r}y^r + \dots + {}^nC_{n-1} xy^{n-1} + {}^nC_n y^n$$

A Binomial is an algebraic expression with at least two terms. It is the **Binomial Expansion** an important formula, which gives out the expansion of power of sums.

We may also write Binomial Expansion as:

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

where n! denotes the factorial of n and for natural numbers n and k, the binomial coefficient can be defined as the coefficient of the monomial x^k in the expansion of $(1+x)^n$.

Arrangement of the coefficients of binomial expansion below shows how binomial expansion reduces to Pingala's MP. {see [4], [13], [17]}

$$\begin{array}{cccccc}
 & & & & & & 1 \\
 & & & & & & & 1 & & 1 \\
 & & & & & & & & 1 & & 2 & & 1 \\
 & & & & & & & & & 1 & & 3 & & 3 & & 1 \\
 & & & & & & & & & & 1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & & & & & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & & & & & & & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1
 \end{array}$$

Here we notice that the coefficients of binomial expansion are identical to the entries of Pingala's MP. It leads to the fact that coefficients of binomial expansion up to any order can be easily deduced from Pingala's MP. Halayudharefer this triangle as 'Staircase of Mount Meru'. [17] and [26]

'Eka-dvi-adi I-g kriya' is the algorithm found from **Chandaḥ Sūtram of Pingala** which gives the procedure to prepare MP. [11] Construction of MP is generally based on the following recursive formula: [7,14]

$${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$$

In each row of MP extreme first and last entry is always one and remaining entries are the sum of two entries of the previous row.

2. HISTORY OF BINOMIAL EXPANSION

Since ancient times people made extensive use of '**Varnic Meru**' or '**Pascal's Triangle**' for the calculation of coefficients in any expansion. Earliest existence of '**Meru-Prastar**' has been found in the work of *Pingala's Chhand Shastra* in 300 BCE [2]. Several modern mathematical concepts such as Combinatory, Sequence, G.P., Pascal's Triangle and Fibonacci sequence were needed and properly used by Pingala and his contemporaries [6,8]. He derived Meru-Prastar for its use in construction of metres and also described the method of finding the number of combinations obtained by taking ekakasamyoga, dwikasamyoga etc, but it was quite difficult to understand. In 10th century CE *Halayudha* explained its meaning well [14], [17]

In 9th century CE, *Mahaviracharya* had given the general formula for the calculation of binomial coefficient nC_r . [7,13,19]. According to him the number of ways of selecting r objects without regarding the order of selection is as follows:

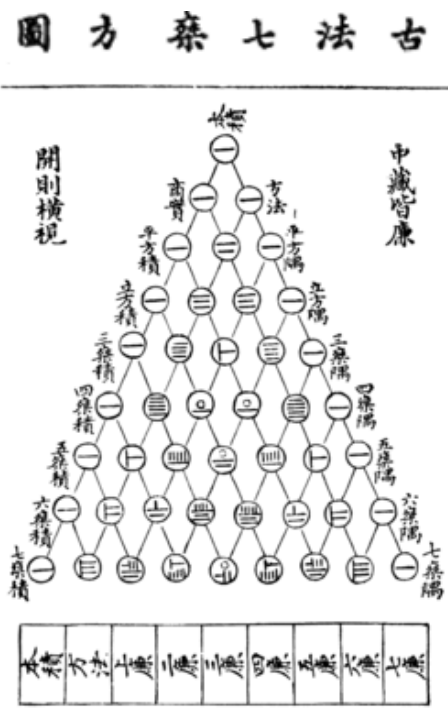
$${}^nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{1,2,3\dots} [19]$$

Jain mathematician called it as *Prastarana* [19] and when the order of selection is taken into consideration then it was called '*Vikalpa*' and denoted by ${}^n P_r$ as

$${}^n P_r = \frac{n!}{(n-r)!}$$

Persian mathematician *Al-Karaji* (c. 953 CE) knew a more general binomial theorem and triangular arrangement for obtaining binomial coefficient in 10th century CE. He also provided a mathematical proof of binomial theorem and Pascal’s triangle using mathematical induction. [21]
 In 12th century CE, *Bhaskaracharya* (c.1150 CE) also had given arithmetical triangle for the calculation of binomial coefficient and named it as *Khandmeru*. It is similar to the triangular arrangement of ‘Meru-Prastar’ given by Pingala. [12]

In 13th century CE Chinese mathematician *Yang Hui* (b.1238-1298 BCE) provided Pascal’s triangle using rod numerals as described in the publications of *Zhu Shijien* in 1303 CE (as shown below) [25]



Yang Hui triangle (Pascal's triangle) using rod numerals as depicted in a publication of Zhu Shijie in 1303 AD

In Yang’s book *XiangjieJiuzhangSuanfa* of 1261 CE there is Chinese illustration of Pascal’s triangle in which Yang acknowledge that his method of finding square roots and cubic roots using Yang Hui’s Triangle was invented and explained by mathematician *Jia Xian* in around 1100 CE, about 500 years before Pascal [23]. In his book *RujiShisuo*, Jia described the method used as ‘li chengshisuo’ (the tabulation system for unlocking binomial coefficients), which is known through his contemporary mathematician *Liu Ruxie*.

In 15th century CE *Cardon* (c.1539 CE), *Michel Stifel* (c.1542 CE) and *Stevin* (c. 1544 CE) also used triangular arrangement for expansion purpose in Europe. [14]

In 16th century CE, this triangular arrangement was rediscovered by French mathematician *Blasé Pascal* and termed as *Pascal’s Triangle*. The coefficient of binomial expansion for all degrees up to any particular positive integer n may be obtained by using triangle.

This triangle is based on the following recurrence relation [25]

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

This relation is known as *Pascal's rule*.

After Pascal a European mathematician *Issac Newton* (1676 CE) in 17th century CE also had given the general formula of the expansion (without proof). Newton was the first who showed that expansion also holds for negative integer (-n) and fractional n as

$$(x+a)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k a^{-n-k}$$

which is *negative binomial series* and converges for $|x| < a$. [24]

One of the simplest Newton series is named after his name and known as *Newton's Binomial series* which is as follows: [24]

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n = 1 + \binom{\alpha}{1} z + \binom{\alpha}{2} z^2 + \dots$$

Besides Newton, Gregory and Leibnitz also began their work with interpolation and attained two of their important results, the Binomial Theorem and Taylor's Theorem. In 1624 Briggs composed 'Briggs Arithmetic aLogarithmica', which is concerned with the calculation of logarithm, uses series for interpolation and during this procedure he gives the first instance of the binomial theorem for a fractional exponent as below:

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{1.1}{2.4} x^2 + \frac{1.3}{2.4} x^3 - \frac{1.1.3.5}{2.4.6.8} x^4 + \dots$$

In eighteenth century, CE *Jacob Bernoulli* gave the proof of negative binomial series. But this proof was published in 1713 CE after his death and *Euler* (1707-1783 CE) also worked on this theorem. *Maclaurin* (c.1742 CE) proved binomial expansion for rational n and *Kastner* (c.1745 CE) proved this expansion for integral values [14].

3. DIFFERENT FORMS OF BINOMIAL EXPANSION

Earlier binomial expansion was known only for positive exponents but in 17th century CE Newton generalized the binomial expansion to negative and fractional n for the first time in the history of mathematical sciences.

For negative n the expansion is as follows: [19]

$$(x+y)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k y^{-n-k}$$

$$\text{and } (1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots \infty$$

In case of positive n the expansion terminates after (n+1) terms but in case of negative n it does not terminate as clear from above example.

4. LIMITATIONS OF BINOMIAL EXPANSION

From previously discussed matter we have seen that binomial expansion for any positive exponent automatically terminates precisely after (n+1) terms. But when exponent is any negative integer, the expansion does not terminate i.e., expansion tends to infinity [17]. For example:

$$(x+y)^{-1} = y^{-1} - y^{-2}x + y^{-3}x^2 - y^{-4}x^3 + \dots \infty$$

(1) When $y = 1$

$$(x+y)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots \infty$$

In this expansion when x equals one then

$$2^{-1} = 1 - 1 + 1 - 1 + 1 \dots$$

$$= (-1)^n$$

(2) When $y = -1$

$$(x - 1)^{-1} = x^{-1} + x^{-2} + x^{-3} + x^{-4} + \dots$$

$$= \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots$$

(3) When $y = 1$ but $x = -1$ then $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$ for $x = -2$

$$(1+2)^{-1} = 1+2+2^2+2^3+\dots \text{ For } |x| < -1.$$

These tests assert that in first case when both terms of the expansion (i.e. x and y) are positive but exponent is negative then the binomial series turned into infinite geometrical series. In next case when y and exponent both are negative (x is positive) the binomial series turned into infinite geometrical series and hence it becomes a special case of binomial expansion. In last case when x and exponent both are negative (y is positive) the expansion gives obscure results. This explanation manifests that binomial expansion for negative exponent is valid only for particular values of x , y and exponent n .

5. MULTINOMIAL EXPANSION

The Binomial theorem can be generalized to include power of sums for more than two terms. Jacques Bernoulli discovered the generalization of this binomial series for more than two terms (1667 CE) and Leibnitz (1648 CE) in seventeenth century and was termed as multinomial expansion. [14]

It can represent as follows:

$$(x_1 + x_2 + x_3 + \dots + x_n)^n = \sum_{k_1, k_2, k_3, \dots, k_m} \binom{n}{k_1, k_2, \dots, k_m} x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$$

where

$$\sum_{i=1}^m k_i = n$$

or

$$k_1 + k_2 + k_3 + \dots + k_m = n$$

and the summation \sum is taken over all sequences of non-negative integer indices k_1 through k_m .

The coefficients are known as *multinomial coefficients* and can be given by the following formula [24]

$$\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! k_3! \dots k_m!}$$

This expansion is valid only for positive integral power and can not be applied to negative and fractional n . as expansion of $(x+y+z+t)^{-2}$ and $(x+y+z+t)^{1/2}$ can not be found by using this formula. [15]

6. PROPOSED CONDITION OF CONVERGENCE FOR MULTINOMIAL EXPANSION

Different conditions of convergence of Binomial Expansion has been discussed [14, 15].

Here we discuss the condition of convergence for Multiple Expansion.

Take multinomial expansion for $(z - x - y)^n$

If $z=1$ we have $[1-(x+y)]^n$

In case of negative exponent i.e. $n = -1$

$$[1-(x+y)]^{-1} = (x+y)^{-1} + (x+y)^{-2} + (x+y)^{-3} + (x+y)^{-4} + \dots$$

This is an infinite geometrical series whose sum tends to

$$s_n = \frac{(x+y)^n - 1}{x+y-1} \text{ when } (x+y) > 1$$

$$= \frac{1-(x+y)^n}{1-x+y} \text{ when } (x+y) < 1$$

Now different conditions are possible:

(a) When $(x+y) < 1$ i.e. $x < 1$ and $y < 1$

$$\begin{aligned} S_n &= \frac{1-(x+y)^n}{1-(x+y)} \\ &= \frac{1-x^n - nx^{n-1}y - \frac{n(n-1)}{2}x^{n-2}y^2 - \dots - y^n}{1-(x+y)} \\ &= \frac{1+n\frac{y}{x} + \frac{n(n-1)}{2}\frac{y^2}{x^2} + \dots + \frac{y^n}{x^n} - \frac{1}{x^n}}{\frac{(x+y)}{x^n} - \frac{1}{x^n}} \text{ dividing by } x^n \end{aligned}$$

For $n \rightarrow \infty, x^n \rightarrow 0$ as $x < 1$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} S_n = \frac{1+0}{\frac{1}{\infty} - \frac{1}{\infty}} = \frac{1}{0} = \infty \text{ (Indeterminate form)}$$

The series is not convergent. Hence the condition is different in case of Multinomial Expansion. In case of binomial expansion when $x < 1$ the series was convergent while in case of multinomial expansion it is not convergent.

(b) When $(x+y) > 1$ i.e. $x > 1$ and $y \rightarrow 0$

$$\begin{aligned} \lim_{x \rightarrow \infty} S_n &= \frac{(x+y)^n - 1}{(x+y) - 1} \\ &= \frac{x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \dots + y^n - 1}{(x+y) - 1} \\ &= \frac{1+n\frac{y}{x} + \frac{n(n-1)}{2}\frac{y^2}{x^2} + \dots + \frac{y^n}{x^n} - \frac{1}{x^n}}{\frac{(x+y)}{x^n} - \frac{1}{x^n}} \text{ dividing by } x^n \end{aligned}$$

When for $n \rightarrow \infty, x^\infty \rightarrow \infty$ as $x > 1$

We have

$$\lim_{n \rightarrow \infty} S_n = \frac{1+0}{\frac{1}{\infty} - \frac{1}{\infty}} = \frac{1}{0} = \infty$$

The condition will remain same if $y > 1$ and $x \rightarrow 0$

In this case also series is divergent.

(c) When $x+y = 1$ i.e. $x < 1$ and $y < 1$

$$\lim_{n \rightarrow \infty} S_n = \frac{1-(x+y)^n}{1-(x+y)} = \frac{1-1^n}{1-1} = \frac{1-1^\infty}{0} \text{ (Indeterminate form)}$$

Hence it is evident that series formed after multinomial expansion is never convergent.

7. APPLICATIONS OF BINOMIAL THEOREM

1. Most important application of Binomial Theorem is that we can expand any power of polynomial by using this theorem which become very difficult by using the method of repeated multiplication.
2. We may also derive the square-root and cube-root of any number by using Binomial Expansion [12].
3. Different cases such as $\sqrt{2} = 1.414$, $\sqrt[3]{2} = 1.26$ etc. can be re-enumerated.
4. From Binomial Theorem we may derive the Leibnitz Theorem.
5. The Binomial Theorem can be utilized in **trigonometry** also. It can be combined with *De Moivre's Formula* to obtain **Multiple-angle-formulas** for the sine and cosine.[21].

On combining Binomial Theorem and De-Moivre's theorem we may find the following values:

$$\cos nx = \sum_{\text{neven}} (-1)^{k/2} \binom{n}{k} \cos^{n-k} x \sin^k x$$

and

$$\sin nx = \sum_{\text{kodd}} (-1)^{\frac{k-1}{2}} \binom{n}{k} \cos^{n-k} x \sin^k x$$

8. CONCLUSION

Importance of Binomial Expansion in modern mathematics cannot be denied. In this paper historical aspects have been discussed in the proposed condition of convergence for Multinomial Expansion. The paper highlights that besides deriving any power of polynomials by using Binomial Expansion we can also derive the square-root, cube-root as well as n-th root of any number by using it. Binomial Expansion on combination with De-Moivre's formula produces the multiple-angle-formula for sine and cosine function.

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